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TILING RECTANGLES AND HALF STRIPS WITH CONGRUENT POLYOMINOES

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1. Introduction

In this paper, we present three new polyominoes that tile rectangles, as well as a new family of polyominoes that tile rectangles. We also give three families of polyominoes, each of which tiles an infinite half strip. All previous examples of polyominoes that tile half strips were either already known to tile a rectangle, or were later found to tile a rectangle. It is still unknown if every polyomino that tiles a half strip also tiles a rectangle.

2. Tiling Rectangles

The question of which polyominoes tile rectangles has attracted a fair amount of attention. We make some definitions.

Definition. A polyomino is *rectifiable* if it tiles a rectangle. The *rectangular order* of a rectifiable polyomino is the smallest number of copies of it which form a rectangle.

In the second edition of Golomb's classic "Polyominoes" [9], several infinite families of rectifiable polyominoes are given, but only nine sporadic examples are known. Curiously, two of these sporadic examples are related by a 2×1 affine transformation.

Figure 1. Two sporadic rectifiable polyominoes

This led us to consider the images of other sporadic rectifiable polyominoes under this tranformation. The following two examples were noticed because they have easy tilings of infinite half strips, but neither has an obvious rectangular tiling.

Figure 2. 2×1 affine tranformations of two other sporadic rectifiable polyominoes

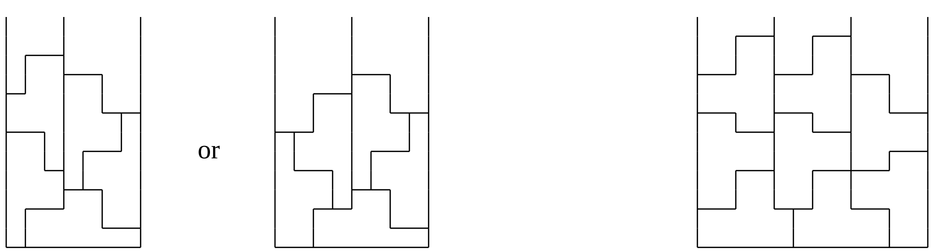


Figure 3. Half strip tilings by these polyominoes

A computer program then found rectangles of sizes 28×132 and 42×230, respectively, for these figures. These are not known to be minimal rectangles, but they are the smallest found to date. The 28×132 rectangle is almost symmetric, except for the central region of 6 tiles. This rectangle has no completely symmetric tiling.

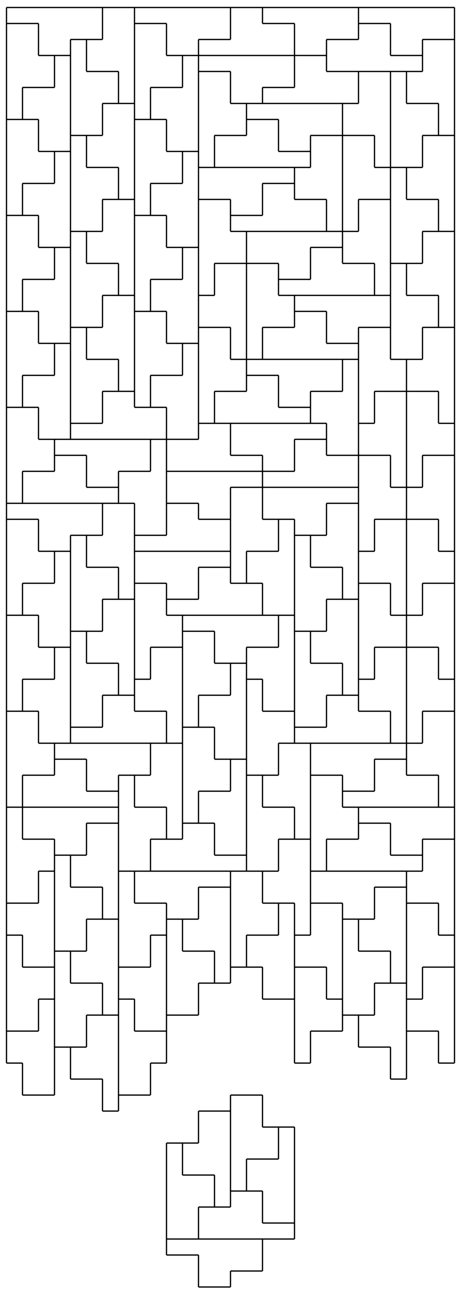


Figure 4. Half of almost symmetric 28×132 rectangle

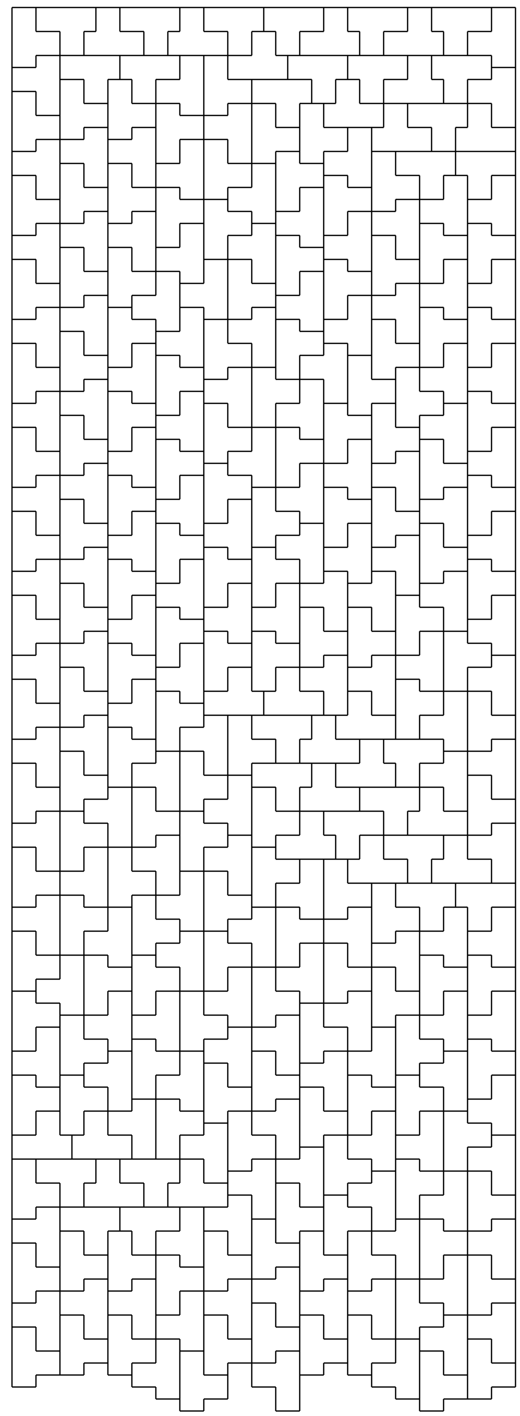


Figure 5. Half of symmetric 42×230 rectangle

Golomb [10] considers the sequence of "boot" polyominoes, notes that several are known to tile rectangles,

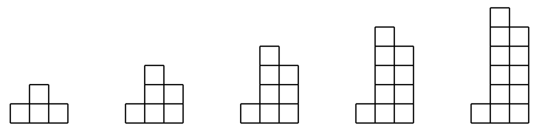


Figure 6. Boot polyominoes

and asks if the others do. The boot tetromino (T tetromino) is well known to have rectangular order 4. Klarner [12] showed that the boot hexomino has rectangular order 18, and Golomb [8] showed that the boot polyomino of size $8n+4$ has rectangular order $4n+4$. Recently, Marshall [10] found that the boot octomino has rectangular order 192 and the boot dekomino has rectangular order 138.

Here we give a partial solution to Golomb's question by showing that the boot polyomino of size $8n+2$ tiles a $(24n+10) \times (112n+28)$ rectangle (Fig. 7), and thus has rectangular order $\leq 336n+140$. It is not known if these are minimal rectangles, except for $n=1$, when it is not minimal.

Description of the Searching Algorithm.

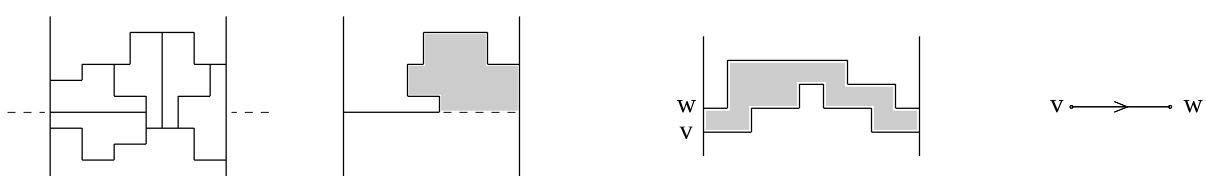
The program that found these tilings was based on the following principle. We give this in the form of a proposition. Part (c) is stated (without details) in [5, p. 184], and the idea is implicit in [1; 6, Theorem 3]. However, it does not appear to have been made explicit.

Proposition 2.1. *Let S be a finite set of polyominoes, and w a fixed width. There is a finite, deterministic algorithm to decide*

(a) *if the set S tiles an infinite (in both directions) strip of width w,*
(b) *if the set S tiles an infinite half strip of width w,*
(c) (Klarner) *if the set S tiles any rectangle of width w, and*
(d) *all lengths l such that S tiles an $l \times w$ rectangle.*

PROOF. Suppose there is an infinite strip tiling. Consider the shape formed by those tiles that cover at least one square below a given grid line. Since S is finite, the polyominoes in S are bounded in length by some M, so the excess part of this shape is contained in a $(M-1) \times w$ rectangle. In particular, there are only finitely many distinct shapes of this ending.

Consider the directed graph G whose vertices correspond to such endings, with an edge from v to w if the region between the ending v at height 0 and the ending w at height 1 can be tiled by S. (Note that this region may be empty.) We also note the special vertex v_0 corresponding to the ending with no excess squares. This graph is finite and can be computed in a finite amount of time.



The tiling questions above can now be rephrased in terms of graph theoretic questions. An infinite strip tiling is equivalent to a infinite (in both directions) directed path. Since G is finite, this is equivalent to a (directed) cycle. Similarly, a half strip tiling is equivalent to a path from the vertex v_0 to a cycle. A rectangular tiling is equivalent to a cycle starting at the vertex v_0.

A breadth-first search starting from a given vertex determines if there is a cycle through that vertex, and if so, the length of the shortest such cycle. Now (a), (b), and (c) follow from finiteness of G. For (d), let l be the shortest length for which S tiles an $l \times w$ rectangle, if one exists. Then for each $1 \leq i < l$, a similar breadth-first search determines the smallest k (if it exists) for which S tiles a $(kl+i) \times w$ rectangle. This completely describes all rectangles of width w tiled by S. Thus (d) is proved. $\square$

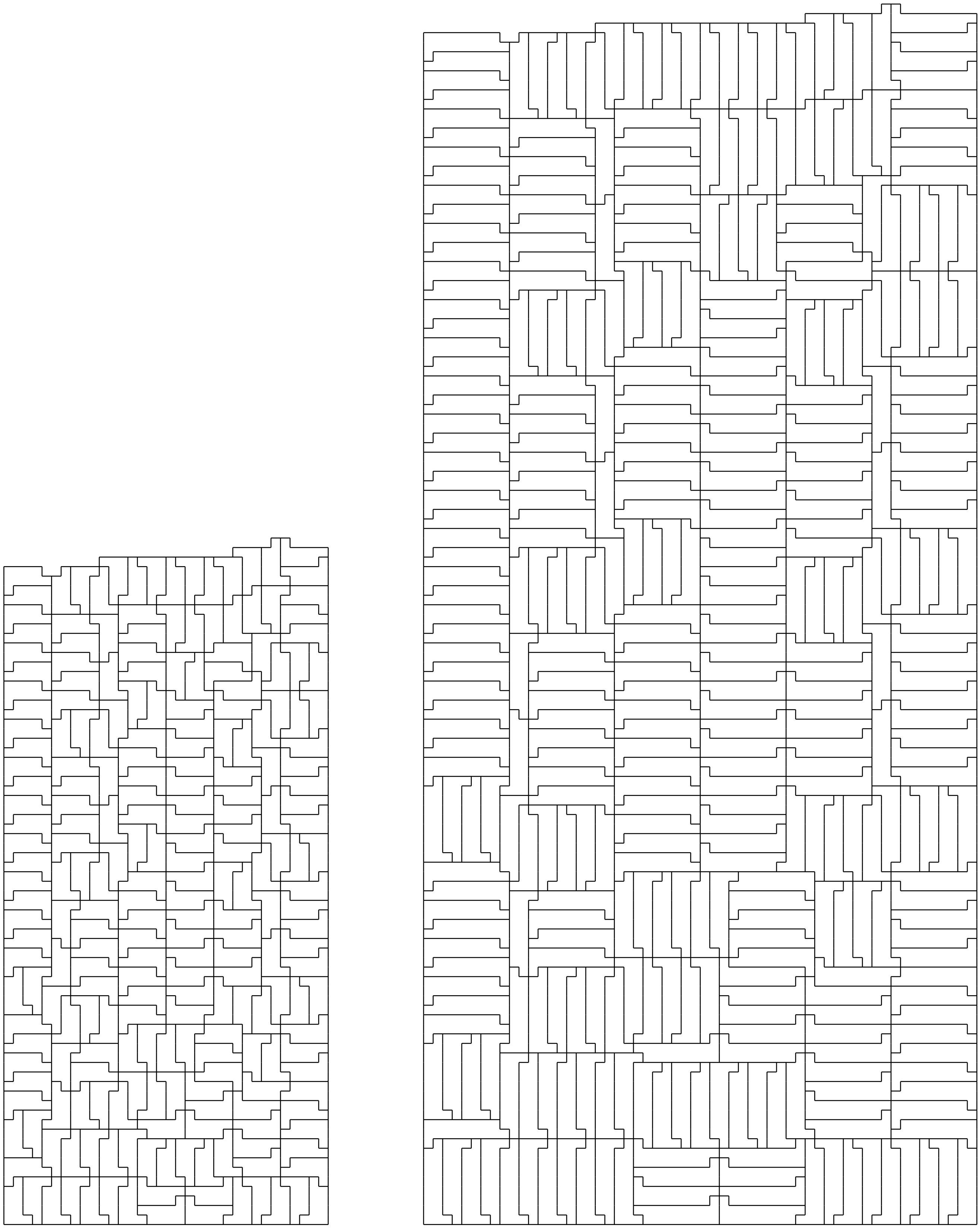
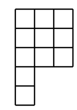


Figure 7. The boot polyomino of size $8n + 2$ tiles a $(24n + 10) \times (112n + 28)$ rectangle

In practice, several enhancements are possible. Most notably, the use of bi-directional search has proved useful. See Bitner [1] for a nice description of his program, where he describes this as forming a rectangle from two partially tiled rectangles.

When using this technique, we tried width $w = 1$ first, then $w = 2$, and so on. Thus we found rectangles in increasing order of their smaller dimension. For the 14-omino of Figure 5, the 42×230 rectangle was the first found, because 42 is the narrowest width that admits a rectangle, and 230 is the shortest length of such a rectangle. The 28×132 rectangle was not the first found for the 12-omino, since it tiles rectangles of widths 26 and 27.

For the boot polyominoes of size $8n + 2$, we first searched for rectangles tiled by the dekomino ($n = 1$). Many rectangles were found, but none appeared to generalize to the rest of the family. We then searched for rectangles tiled by the 18-omino ($n = 2$). The 58×252 rectangle was the first found, and it generalized to all boot polyominoes of size $8n + 2$. It is unclear if every rectangle tiled by this 18-omino also generalizes to the rest of the family.

The same computer program found a 30×154 rectangle tiled by the 11-omino  . This rectangle has no symmetric tiling. It is unknown if this is its smallest rectangular tiling.

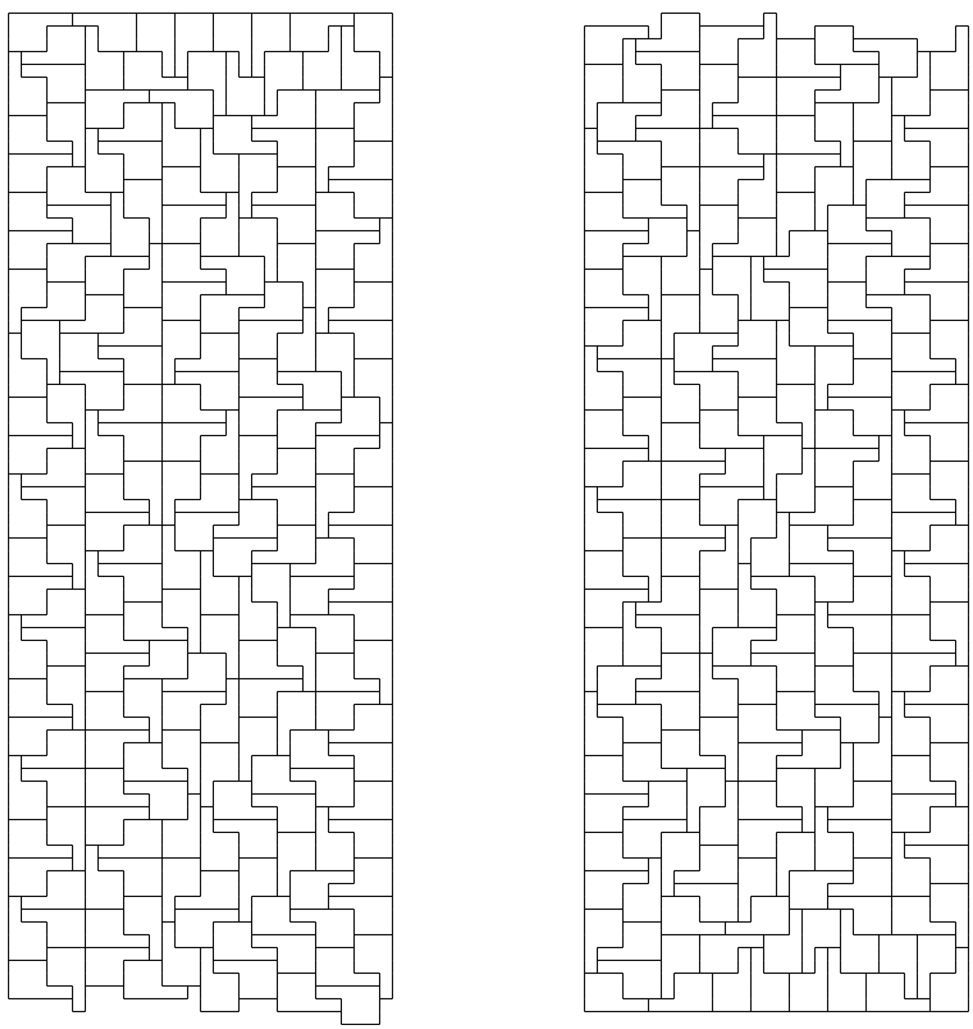
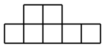


Figure 8. Asymmetric 30×154 rectangle

As another application of this technique, we found all rectangles of width 13 tiled by the heptomino  . These are $13 \times 7n$ for $n = 16, 18$ and all $n \geq 20$.

3. Odd Rectangles

Klarner [13] makes the following definitions.

Definition. A rectifiable polyomino is *odd* if there is some rectangle tiled by an odd number of copies of it; otherwise, it is *even*. The *odd order* of an odd polyomino is the smallest odd number of copies of that polyomino that form a rectangle.

Klarner [13] gives an infinite family of odd polyominoes, each with odd order at most 15. Several other odd polyominoes are known [9, Figure 163; 11; 13]. A second infinite family of odd polyominoes is given in [9, Figure 164]. Here we give a smaller family of odd rectangles for these polyominoes. In particular, the odd order of the L pentomino is 21, not 27 as previously thought [9, 13].

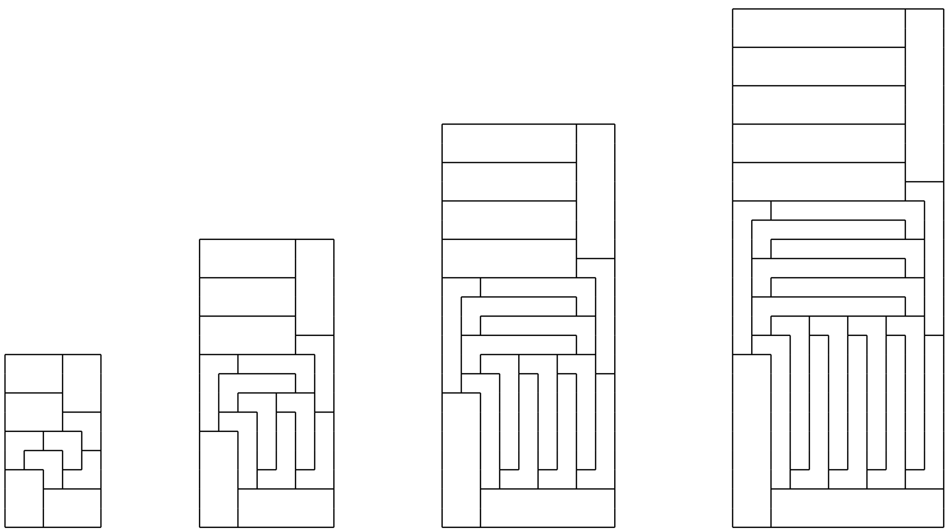
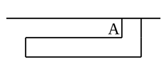


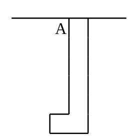
Figure 9. The L n-omino for odd n tiles an $(n + 2) \times 3n$ rectangle

In some cases, we can show that this is the smallest odd rectangle. Let L denote the the n-omino  for some odd $n > 1$.

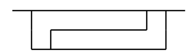
Lemma 3.1. *Suppose that L tiles a rectangle with an odd length edge. Then the long edge of L contributes to this edge. In particular, this odd dimension is at least n.*

Proof. One of the odd length edges of L must contribute to the odd length edge of the rectangle. This gives two cases.

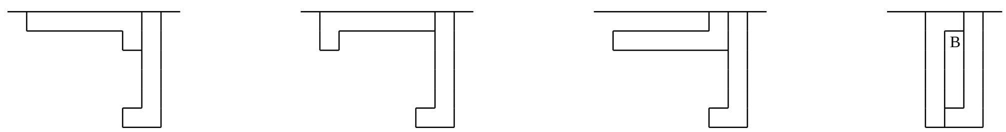
Case 1 

Case 2 

In Case 1, there is only one way to fill square A, and the claim holds.



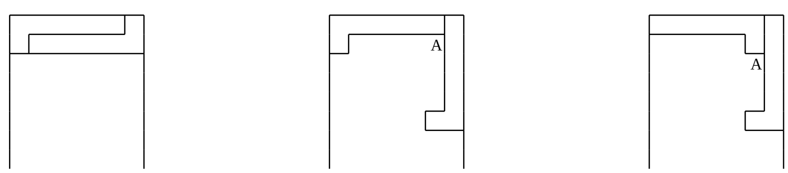
In Case 2, there are four ways to fill square A, as shown. In the first two of these, the claim holds.



The third diagram reverts back to Case 1, and in the fourth diagram, square B cannot be filled. □

Lemma 3.2. *Suppose that L tiles an $m \times n$ rectangle. Then m is even.*

Proof. It suffices to show that the first two rows of such a rectangle are covered by two L's. From Lemma 3.1, there are three ways to cover the first row.



In the first diagram, the claim holds. In the other two, square A cannot be filled without isolating a small region that cannot be tiled by L's. □

Remark. The diagrams implicitly assume that $n > 5$, but the lemma still holds for $n = 3$ and $n = 5$.

Proposition 3.3. *Suppose that n is prime. Then the $(n+2) \times 3n$ rectangle is the unique minimal odd rectangle for L.*

Proof. Suppose that $a \times b$ is a minimal odd rectangle. Then a and b are both odd, so from the lemmas, they are at least $n + 2$. Since n is prime, it divides either a or b, say b, so that $b \geq 3n$. Now the tiling in Figure 9 completes the proof. □

Theorem 3.4. *There is no upper bound to the odd order of polyominoes.*

Proof. For each odd prime p, we've given a polyomino of odd order $3(p + 2)$. □

We conjecture, but have been unable to prove, that the condition that n is prime is unnecessary in the statement of Proposition 3.3.

Another infinite family of odd polyominoes is shown. Specifically, a $2n \times 1$ affine transformation of the P pentomino (in the orientation shown) tiles a $14n \times (30n+5)$ rectangle. This is known to be the minimal odd rectangle for $n = 1$. Figure 10 illustrates these rectangles for $n = 1$ and $n = 2$.

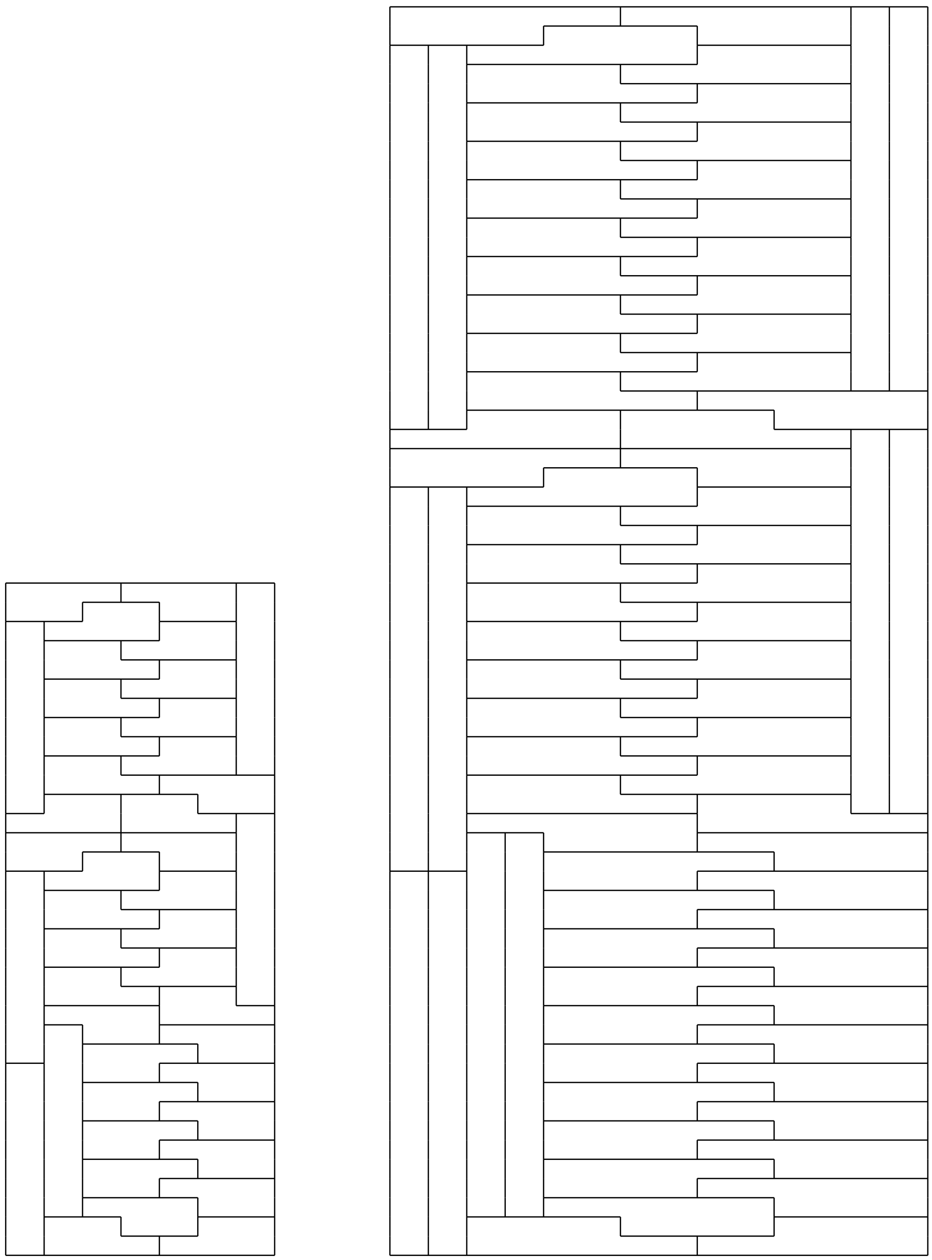


Figure 10. Another infinite family of odd polyominoes

Figure 11 shows several isolated examples of odd polyominoes.

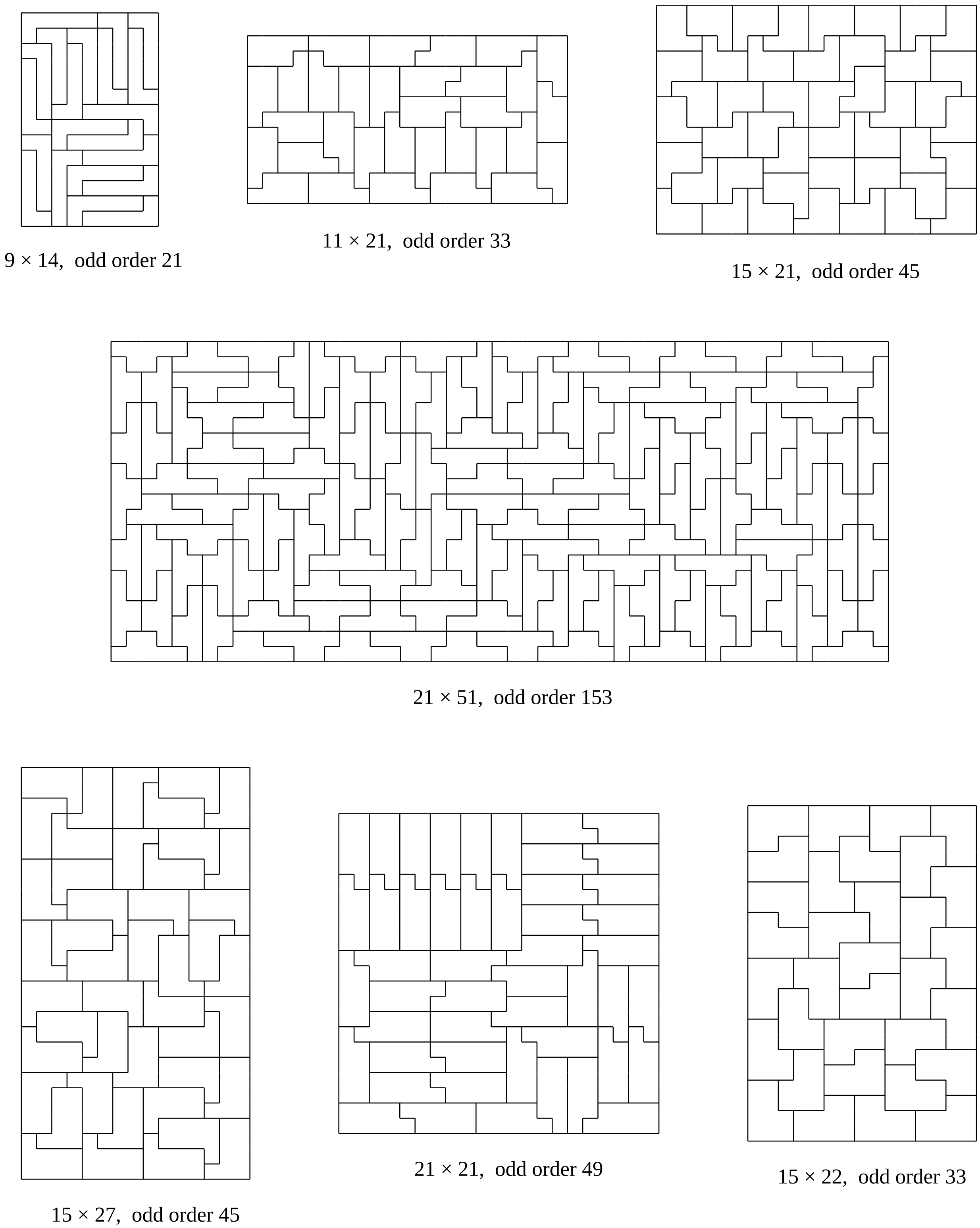
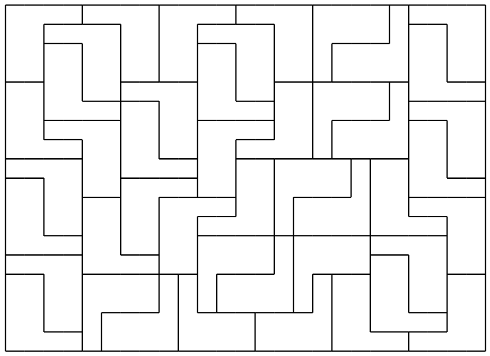
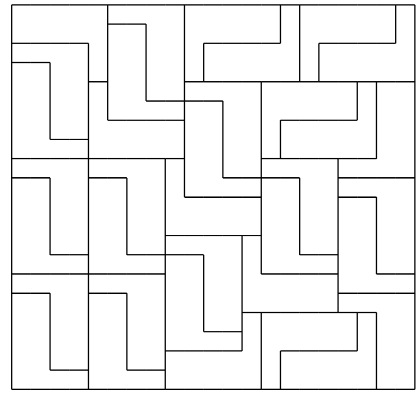


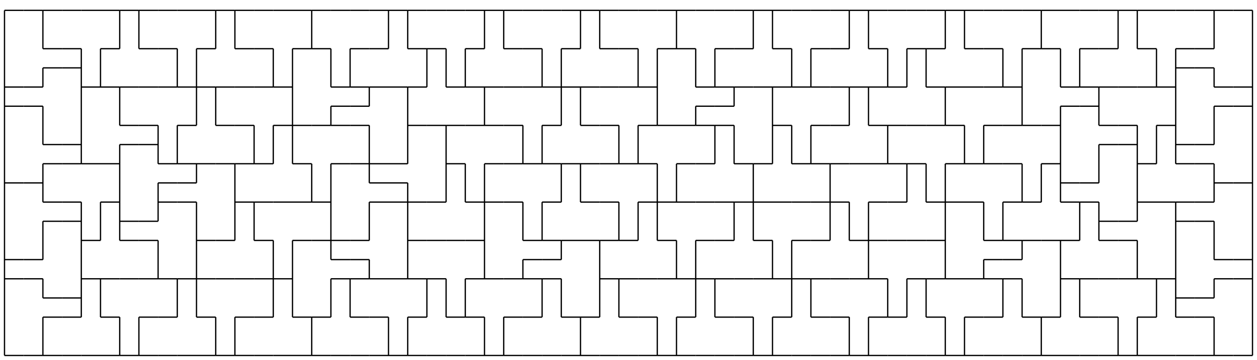
Figure 11. Several odd polyominoes



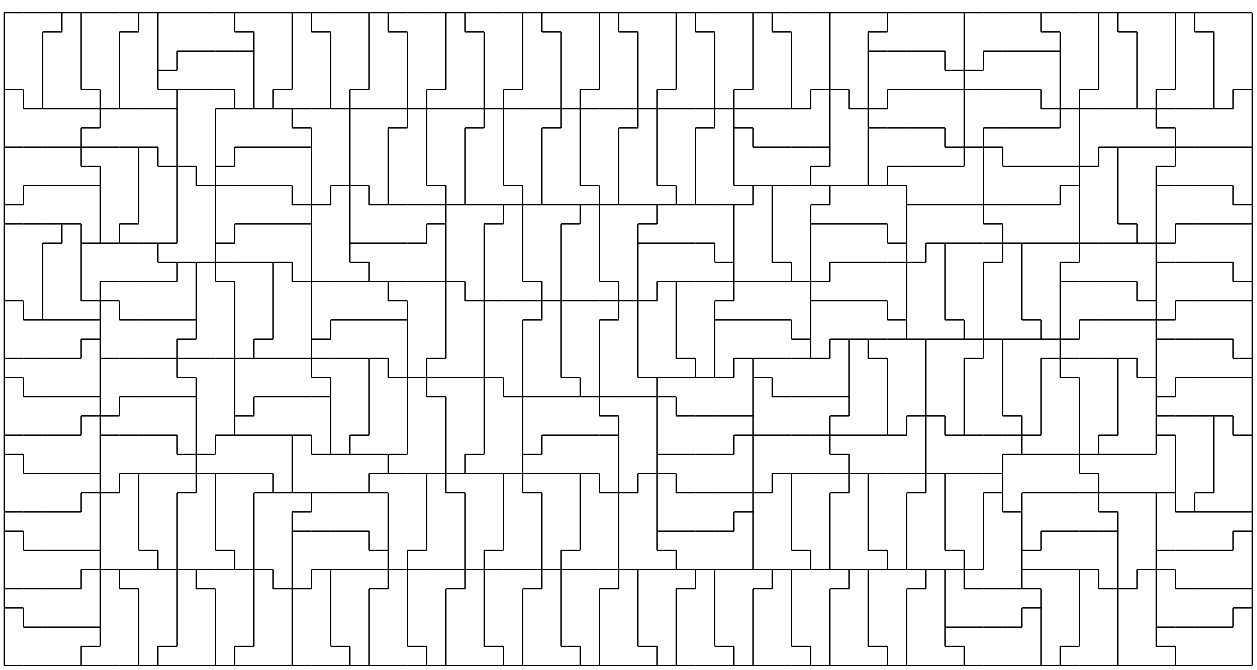
18 × 25, odd order 45



20 × 21, odd order 35



18 × 65, odd order 117



34 × 65, odd order 221

Figure 11 (continued).

More odd polyominoes can be constructed by dissecting an odd polyomino into an odd number of congruent pieces. For example, the polyominoes in Figure 10 can each be dissected into three congruent pieces. Although the resulting figures do not have integer side lengths, we can scale them by a factor of 3 to achieve this. We then get another infinite family of odd polyominoes.

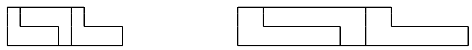
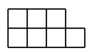
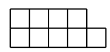
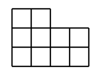
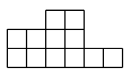


Figure 12. Dissection giving another infinite family of odd polyominoes

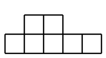
Several more odd polyominoes can be constructed in a similar way. The P pentomino can be dissected into 3 congruent pieces, and the polyominoes  ,  and  can each be dissected into an odd number of congruent pieces.

It seems that perhaps odd polyominoes are more common than originally anticipated. We close this section by exhibiting an even polyomino.

Proposition 3.5. *The* 12*-omino*  *is even.*

Proof. It suffices to show that the 12-omino cannot tile any rectangle with area $\equiv 4 \bmod 8$. Color the square (i, j) black if $i \equiv j \bmod 4$, and white otherwise. Note that any placement of the 12-omino covers either 2 or 4 black squares. Thus any rectangle tiled by this shape covers an even number of black squares. It is an easy exercise to show that any rectangle with area $\equiv 4 \bmod 8$ can be placed so that it covers an odd number of black squares. Therefore any rectangle tiled by the 12-omino must have area divisible by 8, so the 12-omino is even, as claimed. □

4. Tiling Half Strips

The problem of finding polyominoes that tile half strips is also interesting. It is still unknown if every polyomino that tiles a half strip also tiles a rectangle. This is partly because there haven't been many examples of half strip tilers. Previously, every polyomino that was shown to tile a half strip was either already known to, or was later found to tile a rectangle. Furthermore, there were only a handful of examples in the latter category. Golomb [6] showed that the Y hexomino tiles a half strip, and in [7], he showed that the heptomino  also tiles a half strip. It wasn't until much later [2, 3, 14, 15] that rectangular tilings were found for these polyominoes. Taylor [4] gives an octomino with a half strip tiling. Marshall [10] subsequently found a rectangular tiling by this octomino. Wrede [16] gives half strip tilings for three other polyominoes. These polyominoes are now all known to tile rectangles. By mimicking the constructions in Figure 3, we give three infinite families of polyominoes that tile half strips.

Consider the $(32n-4)$-omino shown in Figure 13. Dissect this into two congruent pieces with a path from A to B that is centrally symmetric about its midpoint and alternates between horizontal segments of length 2 and vertical segments of length 1. This path must also begin and end with horizontal segments. For a given n, there are 2^{n-1} possible polyominoes obtained by this construction. These each have half strip tilings with the aperiodic part consisting of two tiles, in a manner analogous to that of the 14-omino of Figure 3. None of these are known to tile rectangles, but none are known not to tile rectangles.

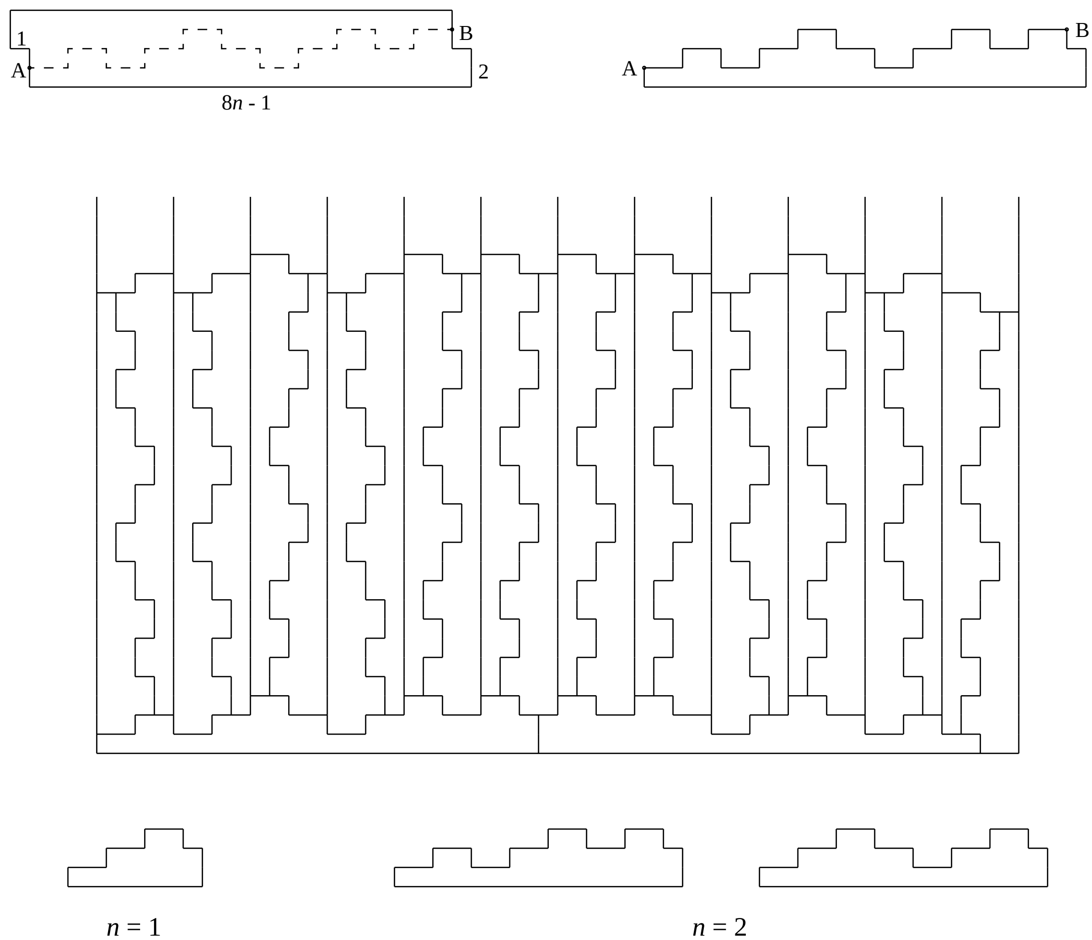


Figure 13. Infinite family of half strip tilers

For a second family, consider the polyomino shown in Figure 14, where k and n are greater than 1. Dissect this into two congruent pieces with a path from A to B that is centrally symmetric about its midpoint and alternates between vertical segments (perhaps of length 0) and k-step staircases. The resulting figure tiles a half strip with the aperiodic part consisting of a single tile. For each pair (k, n), there is exactly one figure formed this way that has an easy rectangular tiling of order 2. For a fixed k, the value $n = 2$ yields only one other figure. However, $n > 2$ allows a continuum of figures. Although most of these figures have sides of non-integer length, those with all edges rational can be scaled to give polyominoes. Apart from the shapes with easy rectangular tilings of order 2, only two figures are known to tile rectangles: the Y pentomino, which occurs as an extremal case for $(k, n) = (2, 3)$, and the 12-omino of Figure 3, which occurs for $(k, n) = (2, 2)$. None of these figures have been shown not to tile a rectangle.

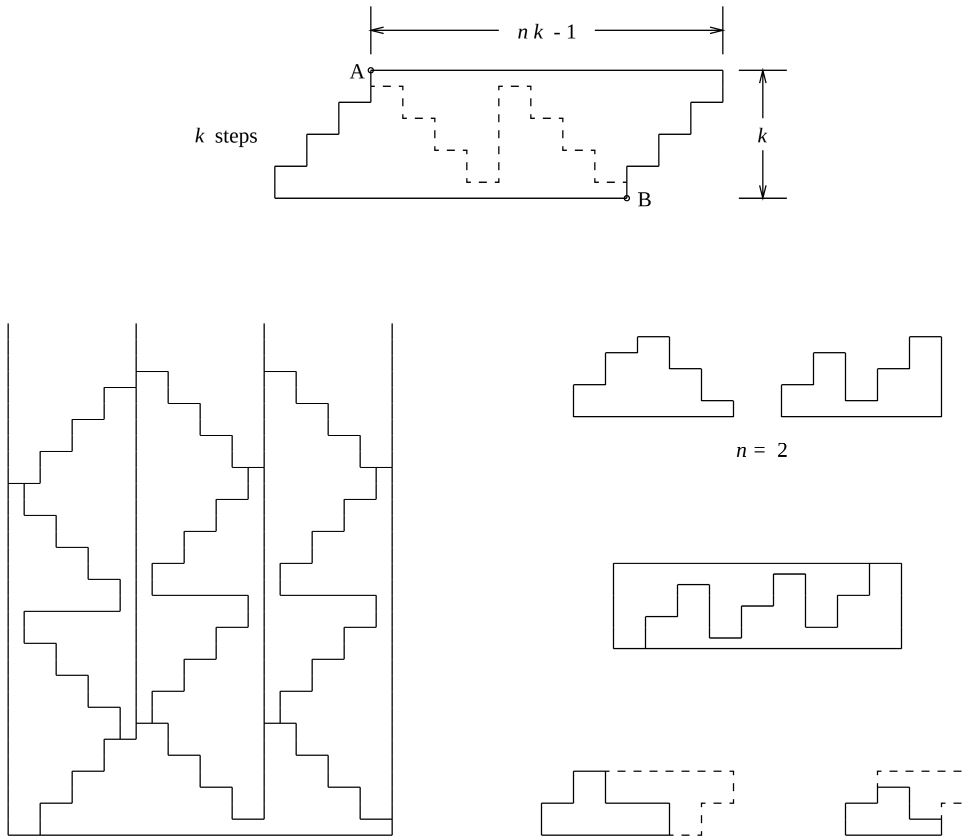
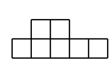


Figure 14. Second family of half strip tilers

A third family is closely related. Dissect the given polyomino into two congruent pieces with a path satisfying the same rules as in the previous family. Again the resulting figure tiles a half strip with the aperiodic part consisting of a single tile. For each pair (k, n), there is exactly one figure formed this way that has an easy rectangular tiling of order 2. For a fixed k, the value $n = 2$ yields only one other figure, but $n > 2$ allows a continuum of figures. These figures can be derived from those in the previous family by adding squares to the steps, as shown. Besides those with easy rectangular tilings of order 2, only one figure in this family is known to tile a rectangle; it is the heptomino  which occurs as an extremal case for $(k, n) = (2, 3)$. None of these figures have been shown not to tile a rectangle.

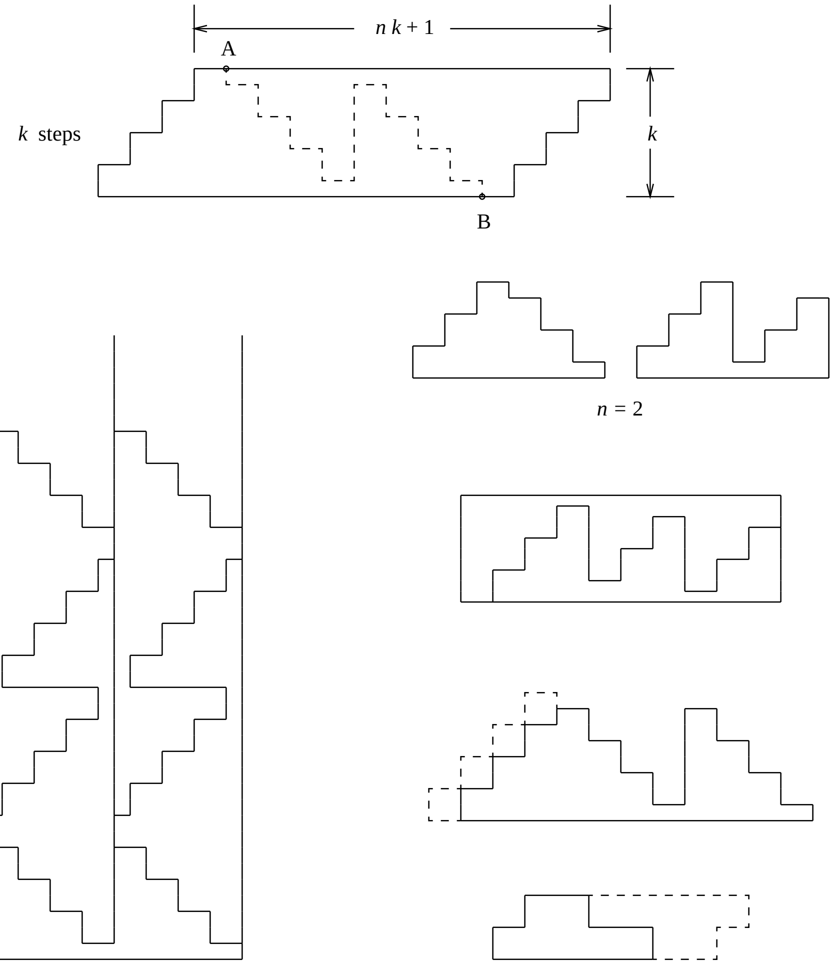


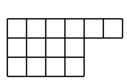
Figure 15. Third family of half strip tilers

5. Further Questions

Question 1. What are the minimal rectangles for the boot polyominoes of size $8n + 2$? (For $n = 1$, the minimal rectangle, 30×46, was found by Marshall [10].) Does every rectangular tiling by the 18-omino generalize to all boot polyominoes of size $8n + 2$? For $n = 1$, the dekomino is odd. Are the larger figures odd?

Question 2. What are the minimal rectangles for the polyominoes in Figures 4, 5 and 8? The 11-omino is known to be odd. Is the 14-omino odd?

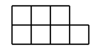
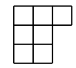
Question 3. Is there a deep explanation for the pairs of sporadic rectifiable polyominoes related by a 2×1 affine transformation? Or is this a case of the "law of small polyominoes"?

Question 4. Can the polyomino  tile a rectangle or even a half strip? This is the only case for images of sporadic rectifiable polyominoes under the 2×1 affine transformation that we could not settle.

Question 5. The odd rectangles of Figure 9 are minimal when n is prime. Are they minimal for (odd) composite n? That the answer is "yes" for $n = 9$ was verified by computer. If n is divisible by 4, it is easy to show that the L n-omino is even. What if $n \equiv 2 \bmod 4$?

Question 6. Which of the half strip tilers can tile a rectangle? Can any of these figures be shown not to tile a rectangle?

Acknowledgments.

The minimal odd rectangles for the heptominoes  and  were found independently by Andrew Clarke (more than 20 years ago), but apparently were never published. The family of odd rectangles in Figure 9 was discovered independently at least three times, by Anton Hanegraaf, by W.R. Marshall and by the present author.

Thanks are due Noam Elkies and Anton Hanegraaf for helpful comments and references. I especially thank Solomon Golomb for his encouragement, and for bringing these problems to my attention. Thanks also to the Mathematics Department at Harvard University for its hospitality, where part of this paper was written.

Addendum.

After this paper was written, the author learned that a number of similar results were obtained independently by William Rex Marshall. See Marshall's paper *Packing Rectangles with Congruent Polyominoes*, J. Combin. Theory Ser. A **77** (1997) pp. 181–92.

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