

TILING WITH SIMILAR POLYOMINOES

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INTRODUCTION

Numerous authors have studied tilings that use congruent copies of a single polyomino shape, for example [1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 14]. In his interesting book [15], Karl Scherer considers tilings that use *similar* copies of a single shape. Here we also consider tilings using similar shapes, but we will restrict our attention to polyomino shapes. We are interested in tiling a rectangle with a given polyomino, and also tiling a larger copy of the polyomino. We will be interested in finding tilings that use the fewest number of pieces. The reason for this is that, from a given tiling, one can derive other tilings that use more pieces.

Following Scherer, we make some definitions. A tiling using similar figures is *regular* if all the tiles have the same size (and thus are congruent); otherwise it is *irregular*. A polyomino is *irrectifiable* if it has a similar tiling of a rectangle. A polyomino is an *irreptile* if it has a similar tiling of a larger copy of itself.

TILING RECTANGLES

Scherer gives an irregular tiling of a rectangle by six copies of the hexomino (Figure 1a). Another rectangular tiling using 6 tiles is also shown (Figure 1b). It has been shown that this hexomino has no regular tiling of a rectangle [16].

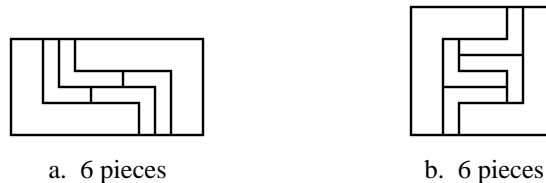


Figure 1.

Figures 2 and 3 show rectangular tilings by two different polyominoes, which each use 8 tiles. The first of these has two different tilings. These two polyominoes each have regular tilings of rectangles, but they require more tiles [7, Figures 6 and 7].

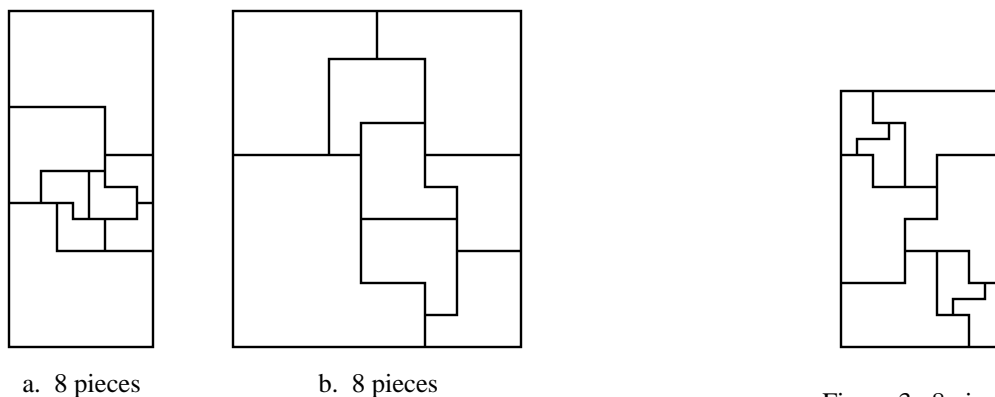


Figure 2.

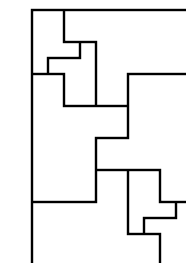


Figure 4 shows several other polyominoes that have irregular rectangular tilings. None of these figures has a regular rectangular tiling.

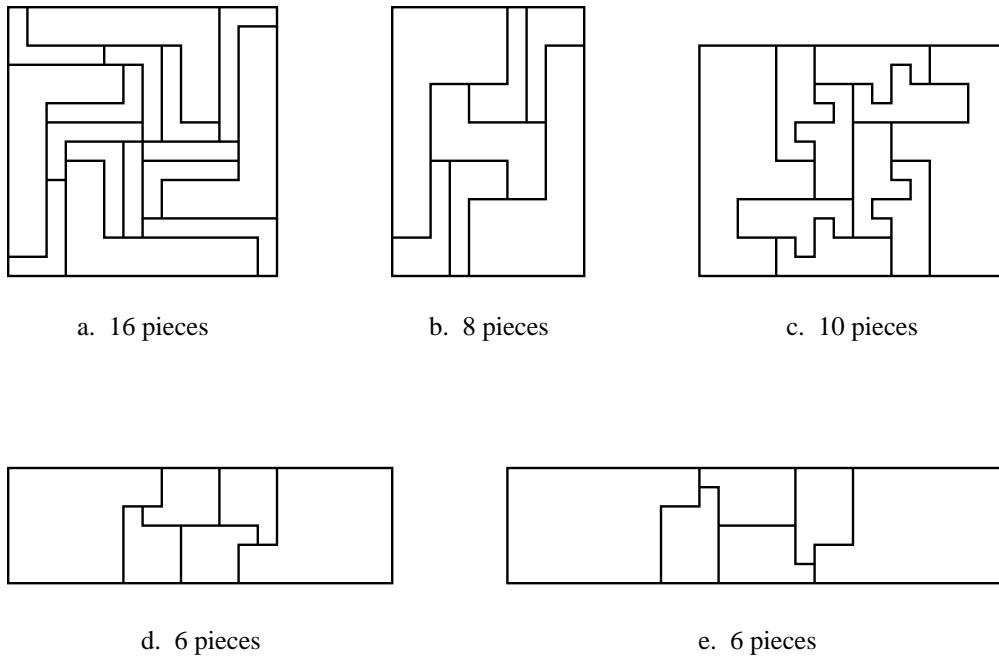


Figure 4. More irrectifiable polyominoes

Several infinite families of irrectifiable polyominoes can be given. In the first family, t is any positive rational number, and the number of steps, n , is arbitrary. Although the edge lengths of the shape might not be integers, we can scale them by a common denominator to make them integral. Then we have an actual polyomino.

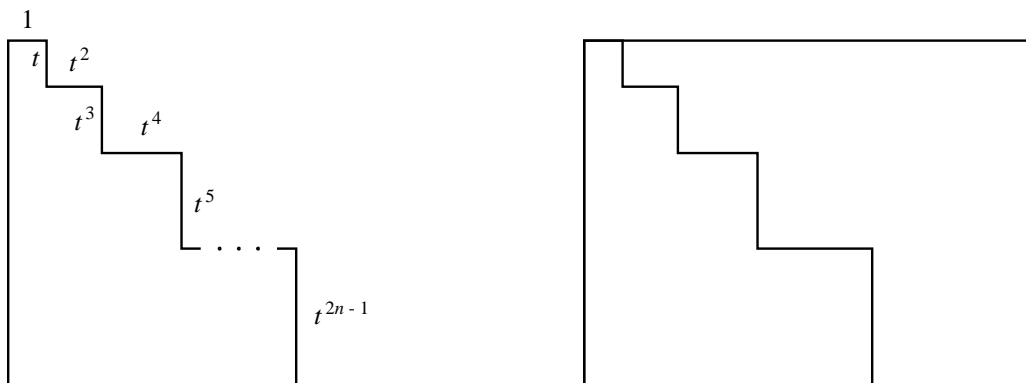


Figure 5.

A second family is closely related. Two staircase shapes that have the same value of t and whose rectangles share a common edge can be juxtaposed to give a new family. There are three combinatorially distinct ways to do this. Note that this also includes the special case where the two staircase shapes are identical; the resulting shape is then symmetric. When $t = 1$, the rectangular tiling is regular, and the shapes are exactly those of Klarner's construction [7, Figure 2].

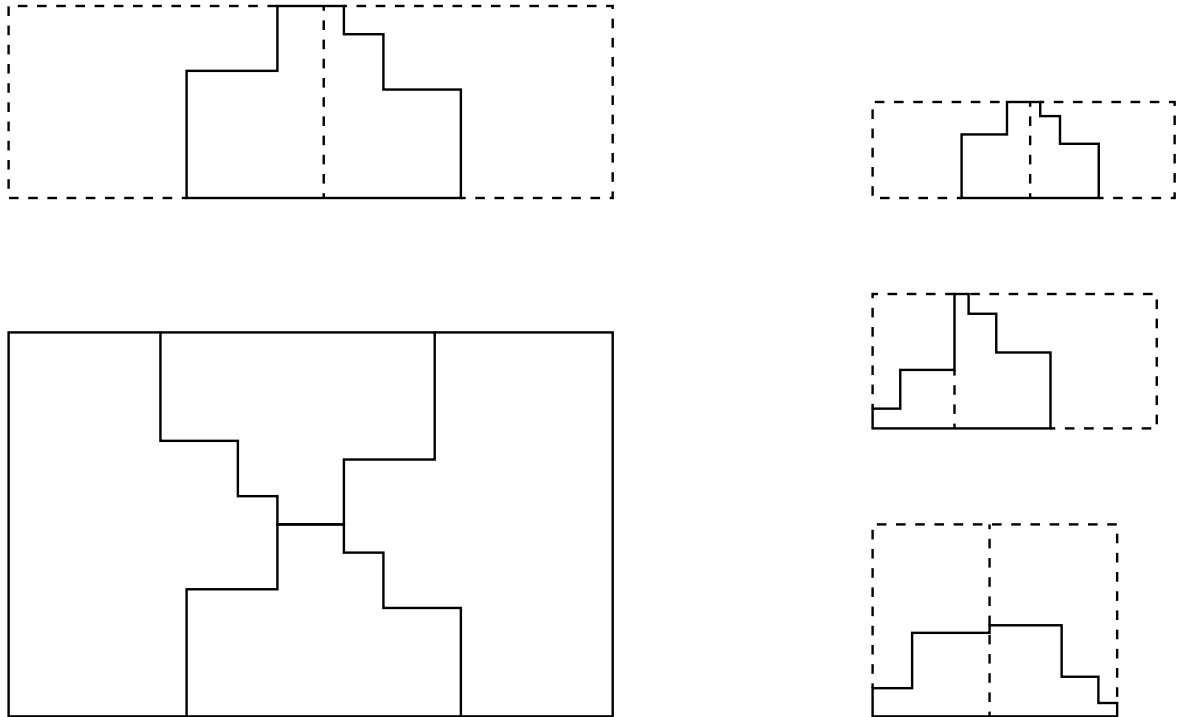


Figure 6.

A third family is shown. Here, t is any rational number greater than $\sqrt{2}$. Figure 7 illustrates for $t = 2$, and the resulting shape is similar to an octomino.

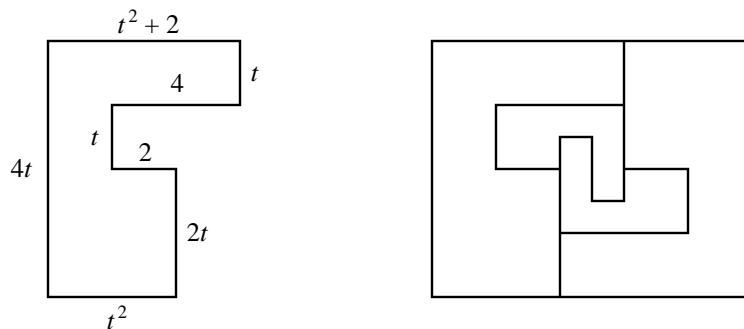


Figure 7.

A fourth family is closely related. Again, t is any rational number greater than $\sqrt{2}$. Figure 8 illustrates for $t = 2$; the resulting shape is (similar to) a dekomino.

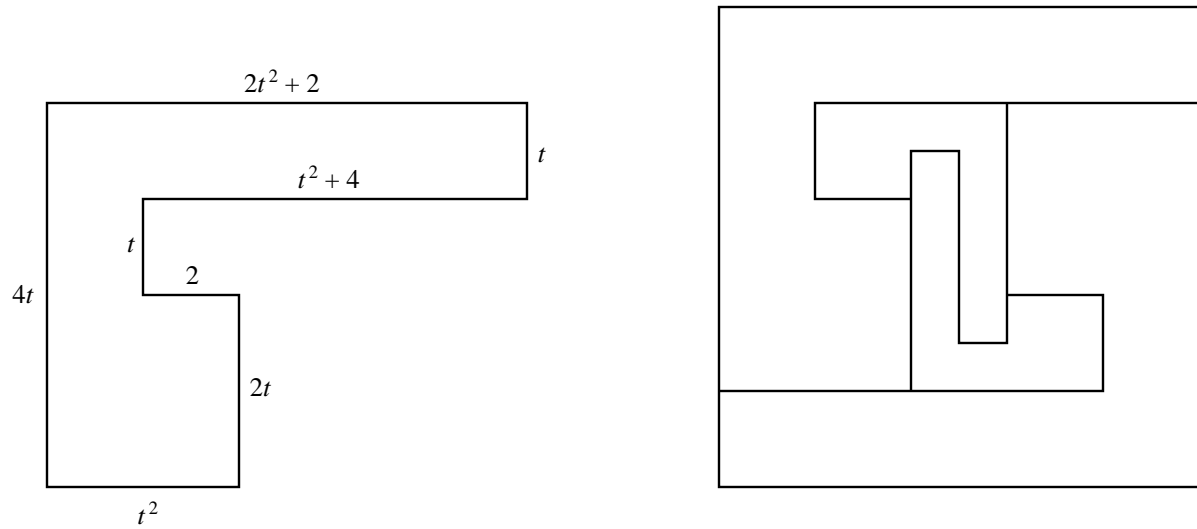


Figure 8.

Another family is given in Figure 9. Here t is any rational number greater than 1. We illustrate for $t = 2$; the resulting shape is a 24-omino.

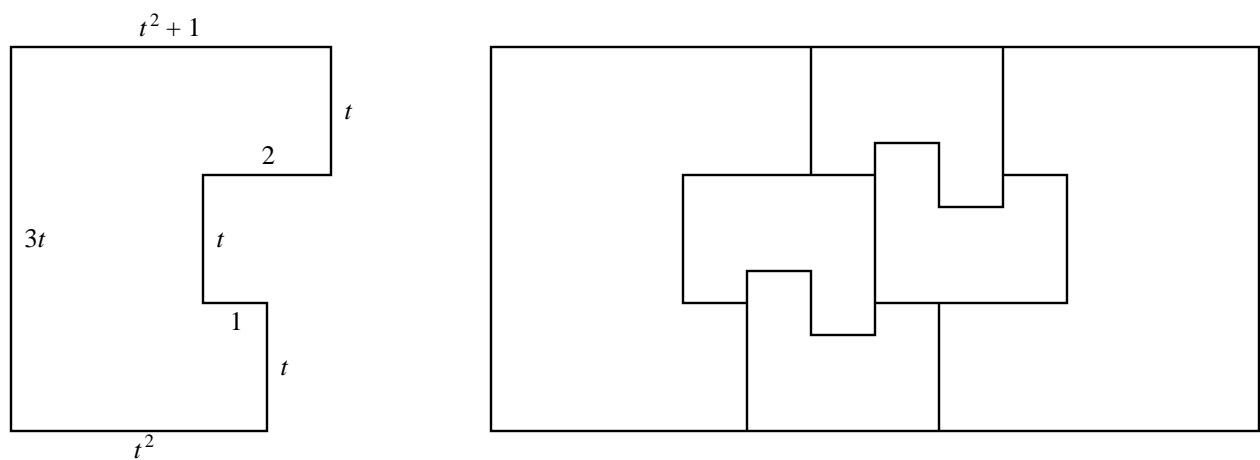


Figure 9.

The most surprising family is based on the heptomino of Figure 2a. Note that the polyomino occurs only in four orientations; it is only reflected horizontally and/or vertically. Therefore the polyomino can be stretched horizontally and/or vertically to make other examples. It would be interesting to see other families based on this type of construction.

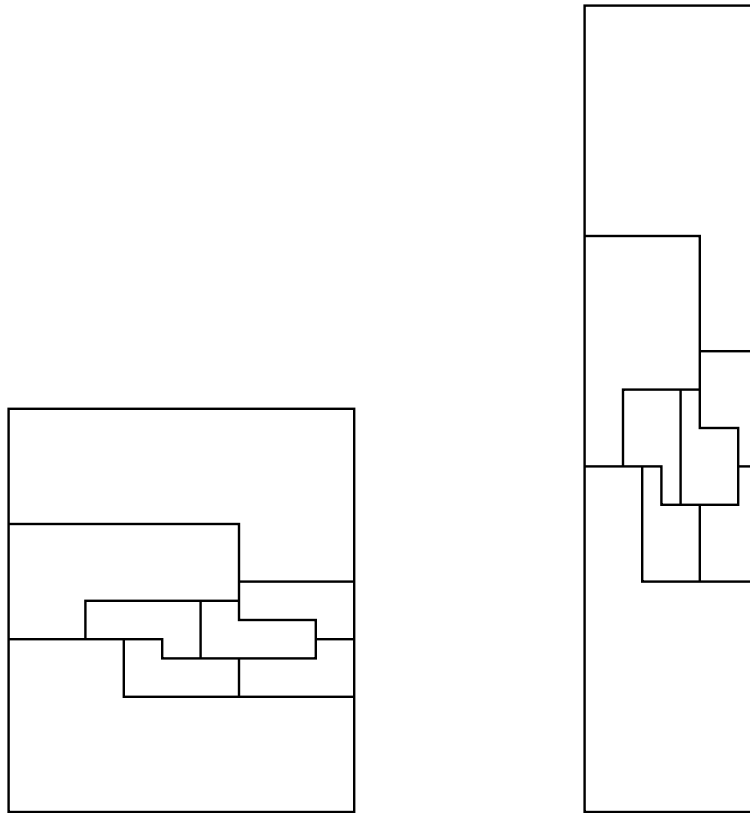


Figure 10.

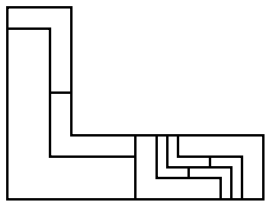
Several other infinite families are given in [11].

IRREPTILES

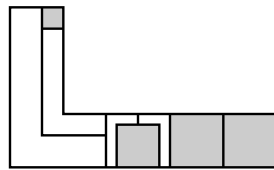
Any irrectifiable polyomino gives an example of a polyomino irreptile. To see this, note that an irrectifiable polyomino has an irregular tiling of a rectangle. Multiple copies of this rectangular tiling can be juxtaposed to give a square tiling, and multiple copies of this form a tiling of a larger copy of the polyomino.

Are there other polyomino irreptiles? None are currently known. However, there seems to be no reason to believe that every polyomino irreptile is irrectifiable, so one should expect that other examples exist. Further investigation into this question is required. (The situation is the same in the realm of regular tilings. Every *known* polyomino reptile is rectifiable, although there seems to be no reason to expect this to hold in general; see [9, Problem 6.10].)

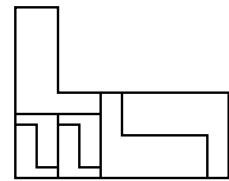
Several examples of irreptilings are shown in Figure 11.



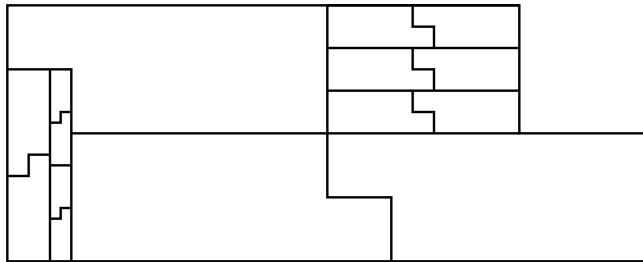
a. 9 pieces



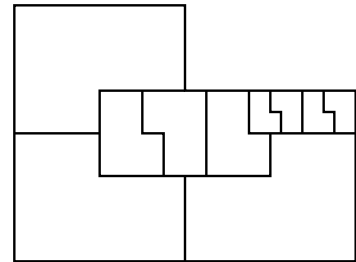
b. 68 pieces (each shaded square is tiled with the 16 piece square of Figure 4a.)



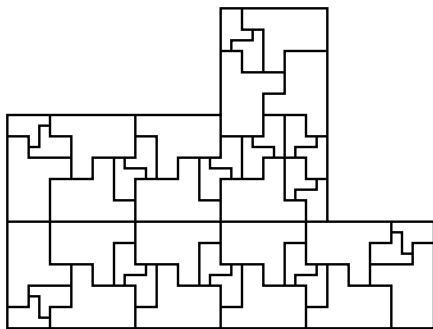
c. 7 pieces



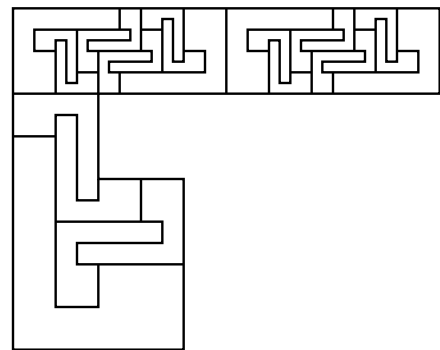
d. 15 pieces



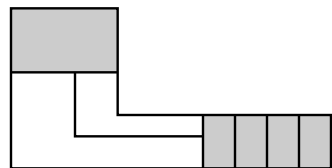
e. 10 pieces



f. 43 pieces



g. 25 pieces



h. 62 pieces (each shaded rectangle is tiled with the 12 piece rectangle shown at right)

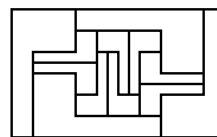
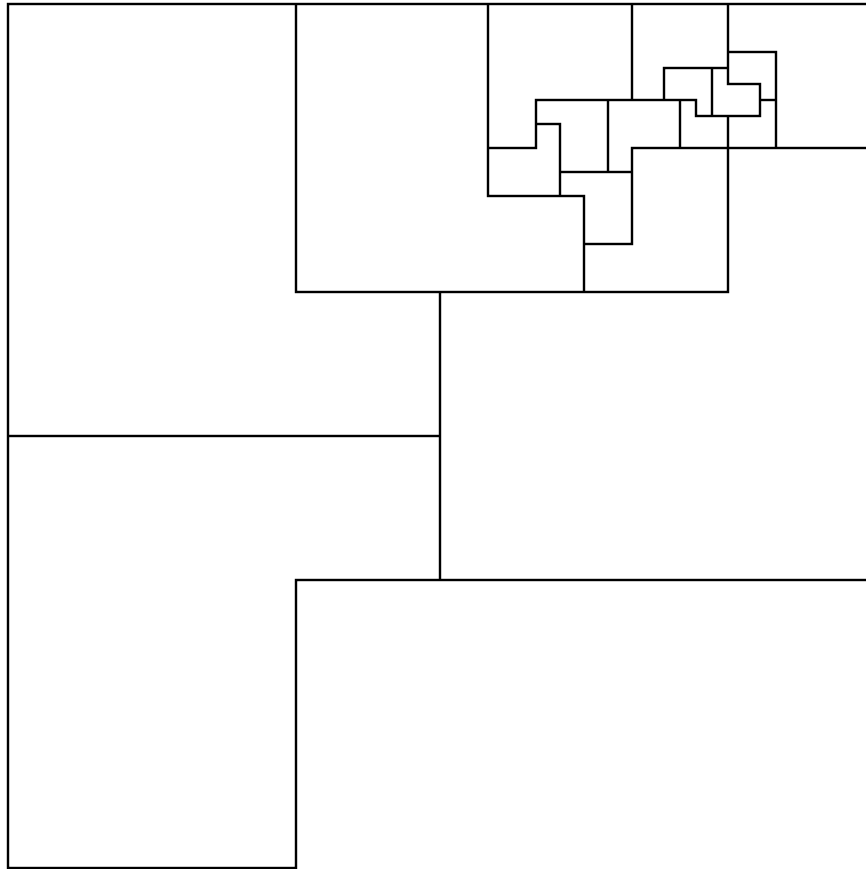
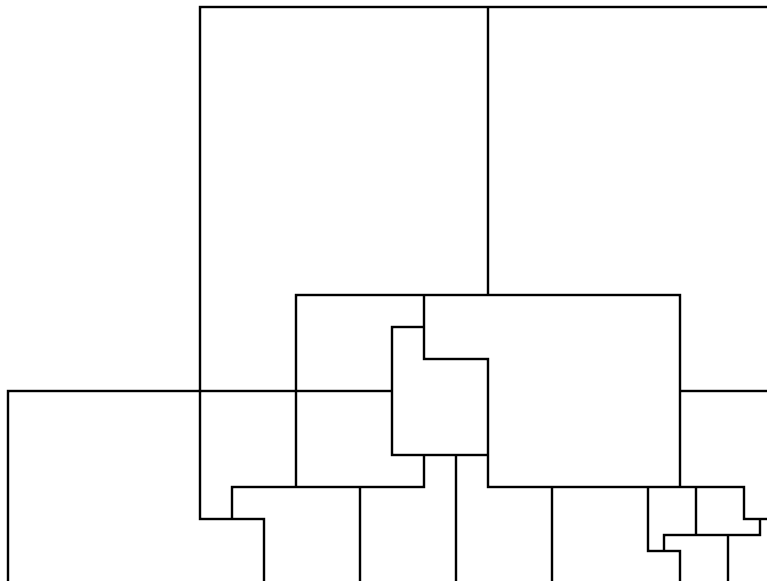


Figure 11. Irreptilings

Some irrectifiable polyominoes have irreptilings that are not built from rectangles. These can be quite striking, as in Figure 12.

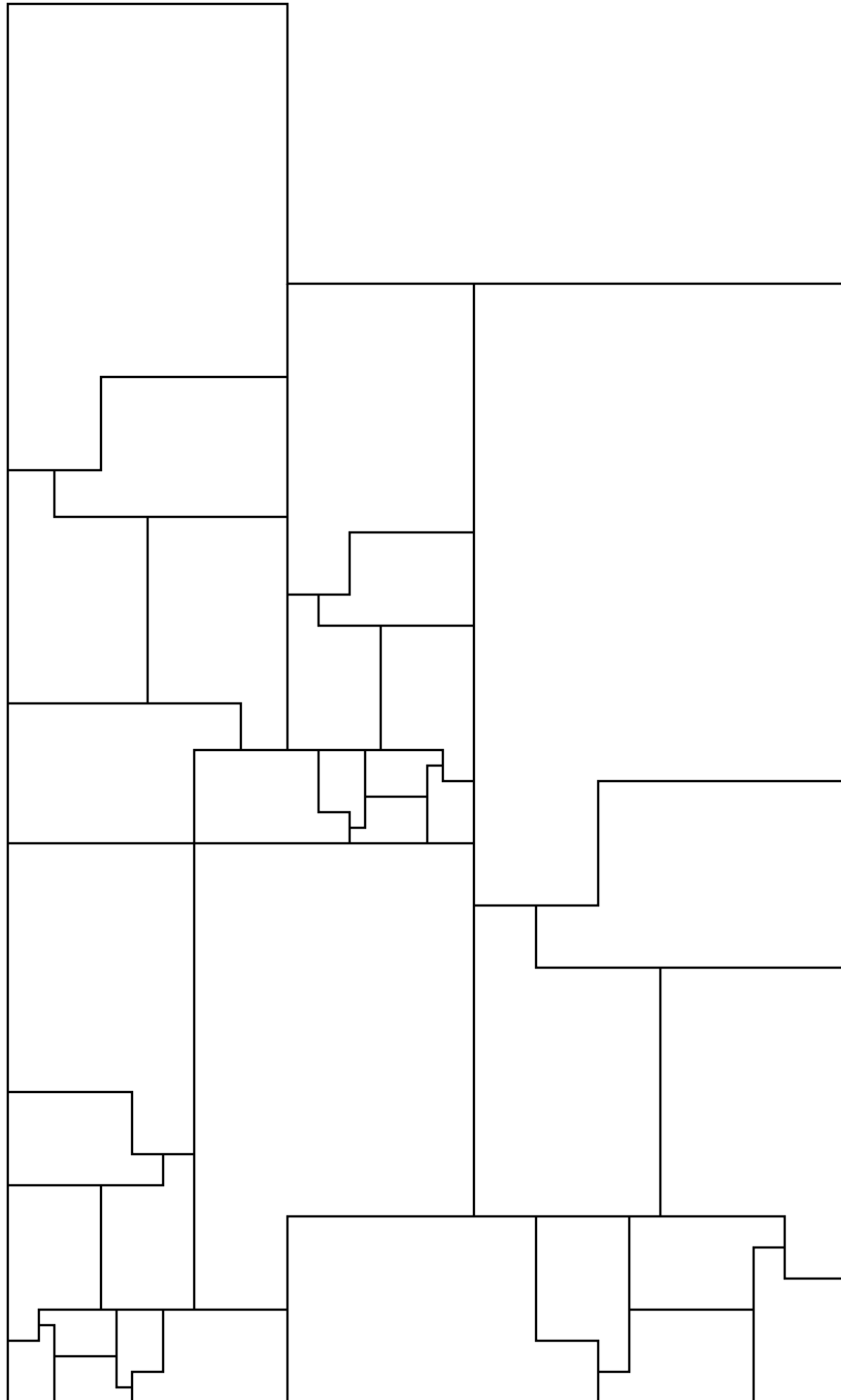


a. 17 pieces



b. 17 pieces

Figure 12 (part 1 of 2). More irreptilings



c. 33 pieces

Figure 12 (part 2 of 2).

More examples of irreptiles can be found in [11].

FURTHER QUESTIONS AND CHALLENGES

We give here some suggestions for further research. We begin with the most obvious task.

- Find more examples of irrectifiable polyominoes and irreptiles.

All of the tilings above were found by pencil and paper. One naturally wonders if a computer program would be effective at finding tilings. In particular, can a computer program determine if these tilings are minimal? The difficulty is that infinitely many different sizes of tiles, different sizes of rectangles and different sizes of the larger copy of the polyomino must be considered. It seems reasonable to believe that only finitely many different sizes can actually occur, with this finite set depending on the shape of the polyomino, and the number of tiles used.

- Devise an algorithm or write a computer program to find irregular tilings. Better yet, write a program that finds tilings that are guaranteed to be minimal.
- Can any of the tilings of this paper be improved, in other words, can tilings be found that use fewer pieces? Can any be shown to be minimal?
- Is there a polyomino irreptile that is not irrectifiable?
- Can other families of irrectifiable polyominoes or irreptiles be found, based on a construction like that of Figure 2a, where the tiles are only reflected horizontally and/or vertically?

ACKNOWLEDGMENTS

This paper was inspired by Karl Scherer's interesting book [15]. Some of the constructions were given in the polyomino bulletin "Puzzle Fun" [11]. The interested reader should consult these two references to see other related tilings.

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